Revisiting Stylized Facts for Modern Stock Markets

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***Abstract*—In 2001, Rama Cont introduced a now-widely used set of 'stylized facts' to synthesize empirical studies of financial time series, resulting in 11 qualitative properties presumed to be universal to all financial markets. Here, we replicate Cont's analyses for a convenience sample of stocks drawn from the U.S. stock market following a fundamental shift in market regulation. Our study relies on the same authoritative data as that used by the U.S. regulator. We find conclusive evidence in the modern market for eight of Cont's original facts, while we find weak support for one additional fact and no support for the remaining two. Our study represents the first test of the original set of 11 stylized facts against a consistent set of stocks, therefore providing insight into how Cont's stylized facts should be viewed in the context of modern stock markets.**

***Index Terms*—stock markets, stylized facts, time series analysis**

I. Introduction

Researchers for decades have sought and proclaimed to have found various 'stylized facts' to characterize the high-level behavior of financial markets. These 'facts' are then used to inform and justify models of these markets. One of, if not the most widely attributed set of stylized facts of financial markets was written by Rama Cont and published in 2001 [15].

Cont presented a list of 11 stylized facts on price variations (returns) in financial markets [15]. Cont's review summarized research ranging over the decades prior, noting seemingly common qualitative characteristics of asset returns across different markets and time frames. It has not been established, however, whether these qualitative characteristics still hold for modern markets and whether they should all be expected to hold for individual stocks for a given time period. In this study, we examine the most granular trading data publicly available over the time period of 18 Oct. 2018 – 19 Mar. 2019, testing whether a stock in the modern market should be expected to express Cont's 11 stylized facts. We find clear support for eight of the stylized facts, with weak support for one other, as summarized in Table I. Section II provides further motivation and background for the study. Section III details our data and methodology for testing the stylized facts. Section IV gives the results of these analyses, and, finally, Section V summarizes takeaways from our results.

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II. Background

Cont's stylized facts [15] drew from numerous past results of Cont and others over the second-half of the 1900's into the early 2000's. Some of the facts were demonstrated through results in the paper, others cited past results, and a couple (detailed in Section II-A) do not appear to have been directly cited or reproduced in the paper. Cont's review has been frequently used to benchmark the empirical relevance of agent-based models to real-world financial markets [6]. Cont himself was involved in these efforts [23], and numerous ABMs have similarly replicated multiple stylized facts [43] [48] [29] [11]. More recently Katahira et al. [30] gave a 'speculation game' model with results reproducing 10 of the 11 facts (all except Fact #3: Gain/Loss Asymmetry). In [10] and [48], the stylized facts were used to determine which parameters produce realistic return series. An assumption implicit in these practices is that most if not all of the stylized facts should hold for a given return series.

Since Cont's set of stylized facts drew from a variety of results and research groups, no single asset (exchange rate, stock, index) was used for all 11 facts [15]. The most facts tested by a single study that we have found is eight, done by Chakraborti et al. [13] in their review on econophysics. They gave details of Facts #1, 2, 4, 5, 6, 7, 8, and 10, providing example results for each of these using intraday returns on the French stock BNPP.PA from 1 Jan. 2007 – 30 May 2008. Determining the extent to which each of the stylized facts should be expected to hold for a given asset is important for understanding how the facts should be used in practice. If any given asset over some time period, which can be viewed as a singe-path realization of a stochastic process, cannot reliably be expected to exhibit all 11 facts, alternative interpretations should be considered. Perhaps stocks on average will exhibit all 11 properties, in which case comparing the expected value of a model's results to the expected value from empirical data could be more appropriate[1].

How the stylized facts are exhibited under different constructions of time is also important to establish. Clock-time, meaning time as measured by a timestamp or date, is most widely represented in the stylized facts literature. As noted by Chakraborti et al. [13], this view inherently involves sampling, and the number of trades in a unit of time can vary widely period-to-period and stock-to-stock (Table II). Event-time, using trades as the event in our case, smooths this variability out, with one trade occurring per unit of time. An event-based view of time is also natural in the context of market simulations, making any differentiation between the stylized facts in clock-time versus event-time crucial to understand when benchmarking these models.

A. Cont's Stylized Facts

Below are Cont's stylized facts as given in [15]:

1) **Absence of autocorrelations:** "(Linear) autocorrelations of asset returns are often insignificant, except for very small intraday timescales ($\approx$ 20 minutes) for which microstructure effects come into play."
2) **Heavy tails:** "The (unconditional) distribution of returns seems to display a power-law or Pareto-like tail, with a tail index which is finite, higher than two and less than five for most data sets studied. In particular this excludes stable laws with infinite variance and the normal distribution. However the precise form of the tails is difficult to determine."
3) **Gain/loss asymmetry:** "One observes large drawdowns in stock prices and stock index values but not equally large upward movements."
4) **Aggregational Gaussianity:** "As one increases the timescale Δt over which returns are calculated, their distribution looks more and more like the normal distribution. In particular, the shape of the distribution is not the same at different timescales."
5) **Intermittency:** "Returns display, at any time scale, a high degree of variability. This is quantified by the presence of irregular bursts in time series of a wide variety of volatility estimators."
6) **Volatility clustering:** "Different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time."
7) **Conditional heavy tails:** "Even after correcting returns for volatility clustering (e.g. via GARCH-type models), the residual time series still exhibit heavy tails. However, the tails are less heavy than in the unconditional distribution of returns."
8) **Slow decay of autocorrelation in absolute returns:** "The autocorrelation function of absolute returns decays slowly as a function of the time lag, roughly as a power law with an exponent $\beta \in [0.2, 0.4]$. This is sometimes interpreted as a sign of long-range dependence."
9) **Leverage effect:** "Most measures of volatility of an asset are negatively correlated with the returns of that asset."
10) **Volume/volatility correlation:** "Trading volume is correlated with all measures of volatility."
11) **Asymmetry in timescales:** "Coarse-grained measures of volatility predict fine-scale volatility better than the other way around."

Financial returns were overall found to be heavy-tailed, not independent and identically distributed (iid), and characterized by correlations and clustering in behavior. Price changes themselves are not claimed to be predicted by any of the stylized facts. Magnitudes of changes, seen as measures of volatility, are found to have nontrivial correlations and relationships with previous behavior. The lack of clear persisting signal on the raw returns is detailed in Fact #1, measured as a lack of linear autocorrelation in returns. This fact is reproduced in the stylized facts paper [15] for event-time returns of the stock KLM and for the USD/Yen exchange rate. Nonzero autocorrelation function (ACF) values at the first lag are found in these and many other intraday results in general, with possible explanations proposed such as the 'bid-ask bounce', nonsynchronous trading effects, and partial price adjustment [3]. The effect is found to decay to roughly zero within 15-minutes by Cont and others in [16] [5] [49].

Some nonlinear transformations of returns, such as taking their absolute or squared values, provide measures of the magnitude of price changes. These volatility measures are found to exhibit persistent positive autocorrelation, in contrast to the linear ACF just discussed. Cont et al. [16] [18] found the absolute 5-minute returns of S&P 500 futures to have ACF values starting above 0.1 and not going below zero for at least 100 lags. Similar results were found in [13] [17] [33] [37] [39]. Explicit power-law fits are given for the decay of autocorrelation in squared and absolute returns in [18] [32] [33]. Power-law decay of absolute autocorrelation implies volatility exhibits long memory or is 'long-range correlated', and we would also in that case expect the correlation to not go to zero or below as the lags increase [36].

The variability of returns is well documented and leads to the second stylized fact: return distributions' heavy tails. Trying to determine the precise distributional form of returns and their tails is a 'favorite pastime' (as Cont put it) in the literature [15] [24] [25] [35]. Consistently agreed upon [1] [2] [9] [13] [15] [16] [18] [39] [41] [42], however, is that returns exhibit kurtosis, the fourth central moment, greater

[1]This is the approach taken by Farmer et al. to validate their 'Zero-Intelligence' model [20]

than that of a normal distribution. Excess kurtosis indicates a distribution has heavier tails and a higher peak than a normal distribution [19]. The financial ABM literature since 2001 has also frequently reported the excess kurtosis of simulated returns to argue the empirical relevance of a model.

In empirical results, returns were found to be leptokurtic for timescales up to multiple days, but kurtosis was found to decrease overall with timescale [1] [13] [16] [18] [42] [39]. This property of 'aggregational Gaussianity' (Fact #4) was shown by Chakraborti et al. [13] to occur more quickly in trade-time than in clock-time, explained as trade-time correcting for some volatility versus clock-time returns. Tail-heaviness decreasing but not necessarily disappearing through methods of volatility correction is summarized in Fact #7, 'conditional heavy tails'. Bollerslev et al. [9] detail the 'kurtosis problem' of fat-tails remaining in the residuals after applying ARCH-type models to stock returns. In [1], Andersen and Bollerslev normalized 5-minute DEM/USD FX returns by an estimate of the average daily volatility pattern and reduced the kurtosis from 21.5 to 15.8. Andersen et al. [2], however, found nearly normal kurtosis values for 5-minute returns of 30 DJIA stocks after normalizing by their respective realized daily variances.

Taken together, returns' heavy tails and volatility clustering lead to the characteristic of returns irregularly displaying periods of high volatility interspersed with long periods of relative calm. This 'intermittency' is Fact #5, and in much of the literature it is discussed as being visibly apparent in the returns series [15] [30] or following directly from these other facts [15]. Cont [15] discusses the multifractal model as a possible explanation of intermittency, suggesting possible multiplicative processes operating across multiple timescales. Arneodo et al. [5] provide evidence of this, particularly arguing for a multiplicative cascade of information from coarse timescales to finer timescales. Müller et al. [38] presented evidence of this 'asymmetry in timescales' effect (Fact #11) in 1997 with a different methodology. They calculated fine volatility as being the average absolute daily return over a given week and coarse volatility as the absolute price change over the full weekly interval. They measured the correlation between fine volatility and lagged coarse volatility with lags τ of -1, 0, and 1, finding the correlation at $\tau = -1$ to be larger than at $\tau = 1$. In similar analysis, Gençay et al. [22] found low volatility at a long timescale was likely to be followed by low volatility at a shorter timescale whereas high volatility did not necessarily show this same 'vertical dependence'.

Two other correlational findings are given by Facts #9 and 10. In #9, the 'leverage effect' expects volatility to be negatively associated with returns. Citing results from Bouchaud et al. [12] and Pagan [40], Cont specifically describes this effect as showing a negative correlation between returns and subsequent squared returns, suggesting negative returns lead to increased volatility. Correlation of volatility with subsequent returns was found to be negligible, meanwhile. Variations of this effect have been noted elsewhere in the literature, however, and negative (or even nonzero) correlation between returns and observed volatility is not always found [7].

In Fact #10, volatility is found to be positively correlated with trading volume. Clark [14] noted this relationship as far back as the 1970's when examining cotton prices. The relationship has been measured over the years through various means, including taking the correlation between shares traded and absolute returns over a period of time [26] and measuring return variance as a function of trades [13] [31] [41] [45]. It has also been proposed that long-range autocorrelation of trading volume leads to volatility clustering [41].

Finally, 'gain/loss asymmetry', the 3rd stylized fact, is perhaps the least clear to interpret from the detail given by Cont [15]. This was summarized as larger drawdowns being seen than upward movements for stock prices and index values. It is possible this is referring to results given in [15] showing negative skewness for S&P 500 futures, Dollar/DM Futures, and Dollar/Swiss Franc futures, each at 5-minute timescales. Skew implies something slightly different from the fact as summarized by Cont, however, as it does not necessarily tell you anything about which tail has larger values. Other studies have also found positive skew rather than negative [44]. Some of the literature since Cont [15] has examined 'gain/loss asymmetry' from another direction, looking at the amount of time it takes to see a gain versus a loss above a certain magnitude. This 'inverse statistic' has been used to show that a stock index will typically achieve a loss more quickly than a gain of the same magnitude [27], but the same property was not found for individual stocks [28], with correlated downward movements across stocks proposed as an explanation for why the phenomenon could arise in indices [46].

B. The National Market System

Finally, it is possible that some of the stylized facts no longer hold much descriptive power due to structural changes to modern markets. The U.S. stock market, known as the National Market System (NMS), has had numerous regulatory and technological changes this century. Trades in the NMS occur through the matching of buyers and sellers of a stock at a given price point. This can occur on stock exchanges or off-exchange through brokers, peer-to-peer trading, or at Alternative Trading Systems (ATSs). As-of 2018, there were 13 stock exchanges split across four geographic locations in northern New Jersey, and three more exchanges have been added by the time of this writing. Roughly 30% of trades in our data occurred at locations other than stock exchanges, such as through broker internalization or on Alternative Trading Systems (ATSs). Messages and trades from this array of venues are consolidated by the three Security Information Processor (SIP) 'tapes'. Exchanges and off-exchange venues must report trades to the SIP tape corresponding to the traded security based on its listing exchange. This is diagrammed at a high level in Fig. 1. Tivnan et al. [47] give a detailed summary of the market's infrastructure circa 2016, with evidence of impact from its fragmentation on the prices acted on by traders.

Each type of trading venue has its own rules and data-reporting requirements, and each venue reports trades with

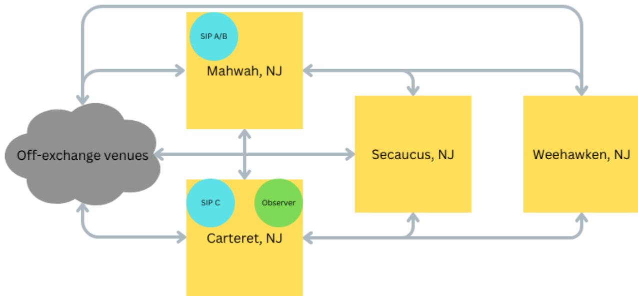


Fig. 1: The U.S. National Market System (NMS) circa 2018. The data collected here was collected from the SIP feeds by Thesys, the 'observer' in Carteret. Image recreated from Van Oort et al. [50]

some (small but nonnegligible) allowed latency[2]. Systems executing automated orders are required to sync their clocks to within 50ms of the time maintained by the National Institute of Standards and Technology (NIST) in the US [34]. This may seem a short amount of time, except the NMS is increasingly operating at speeds approaching the speed of light [4]. Driven by algorithmic trading, a stock price can have an extreme rise or drop and rebound back to nearly its original level in less than a blink of the eye [29]. Andersen et al. [2] noted for their TAQ data on 30 DJIA stocks over the time-span of 1993-1998 a median inter-trade duration of 23.1 seconds on average for the stocks in their sample. For contrast, in our sample, the median inter-trade duration for MSFT was 22μs, a reduction of more than 10^3 in magnitude. Given these modern developments, it is vital to understand whether the high-level qualitative behavior of the market Cont and others observed circa 2001 still holds in the market we have today.

III. Methodology

We test each of Cont's 11 stylized facts on U.S. stock data for the date range 18 Oct. 2018 – 19 Mar. 2019 (103 trading days). We specifically have looked at ten stocks in our current analysis: AAPL, AMZN, BRK.B, JNJ, JPM, MSFT, NVDA, TSLA, V, XOM. This list is somewhat arbitrary but consists of highly traded stocks, providing us with an immense amount of data and detail of how the market behaved over the time period examined. Five of these stocks are listed at Nasdaq, five are listed at NYSE. Our data contains all trades reported for these symbols over the date range. The data was provided by Thesys Group Inc., which acted as a tape consolidator in the NMS and was the sole data provider for the SEC's MIDAS in this time period [47]. The Thesys data was collected in the Nasdaq data center in Carteret, NJ (Fig. 1). Due to the latency and limitations on clock synchronization mentioned in Section II-B, the exact order of trades in the data is not definitive, but our time series provides the perspective of an observer located in Carteret, NJ, viewing the events of the market as reported by the SIPs.

We consider two different views of time when constructing our return time series: clock-time based on timestamp and event time, with trades as the event. In either view, in order to aggregate to any level that is more granular than a single trade per unit of time, we take the price of the last trade to occur in that time period. Let $X(t, \Delta t) = \log P(t, \Delta t)$, the log-price. The log-return at time t and timescale Δt is defined as $r(t, \Delta t) = X(t, \Delta t) - X(t-1, \Delta t)$. Any reference to returns going forward should be interpreted as meaning log-returns. Absolute returns refers to the absolute value of $r(t, \Delta t)$. In clock-time, time points with no trades will be assumed to have the same price as the previous time point that had at least one trade, as no new price information has been received since then. Note that the return at that time point will therefore be 0.

We limit our time series to the trading day (9:30am - 4:00pm ET), filtering out 'after-hours' trading activity. We also filter out the batch auctions that start and end each day. In our price time series then, the last price before the close from one trading day will be followed immediately by the first price after the open of the next trading day. Due to this construction, an overnight return between t_1 = 16:00 on a given day and t_2 = 9:30 the following day is characteristically different from a return between two sequential prices within the same trading day. Therefore, we only consider returns $r(t, \Delta t)$ such that t and $t - \Delta t$ are within the range 9:30 – 16:00 of the same trading day. If the last period of any day is incomplete (e.g. $\Delta t = 50Min$ would result in a 40-minute period at the end of the day), we remove that return from our series. It is worth noting that the number of data points in a day will vary inversely with the timescale. Most stocks will have tens if not hundreds of thousands of trades per day, whereas there are 390 minutes in a 6.5-hour trading day.

Consistent with much of the results and references given by Cont [15], the main tools used in our analysis are correlation, the calculation of moments, and describing the distributions of events. We utilize Pearson sample correlation, denoted as $\text{corr}(x, y)$ going forward, to calculate the correlation values. In order to judge what is a consistent, nontrivial feature of the returns in our sample, we look at the extent a result shows a consistent signal across symbols, with that signal differing from what is observed for randomly generated 'white noise' returns. To generate an instance of white noise returns, we create 103 days of prices whose trade-level returns are iid, normally distributed ($N(0, 0.000696)$), with 250,000 trades randomly (uniformly) distributed throughout each day. By generating 100 of these 103-day price series and calculating results for each, we get a threshold of what type of behavior arises from random white noise.

IV. Results

Before getting into the specific results for each fact, Table I gives a high-level summary of our findings. We find clear evidence for eight of the 11 and weak evidence for Fact #11.

[2]Bartlett and McCrary [8] found trades of DOW30 stocks to be processed by the SIPs 24ms on average after they were recorded by exchange matching-engines over the period of 6 Aug. 2015 – 30 Jun. 2016. Off-exchange trades, meanwhile, are allowed up to 10 seconds to be reported to Trade Reporting Facilities (TRFs) which then report the trades to the SIPs [21].

TABLE I: Breakdown of which facts we found evidence for in clock-time and event-time. 'X' marks indicate strong evidence found for a fact, while '∼' indicates that only partial or weak evidence was found.

Fact #	Fact Name	Clock-time	Event-time
1	Lack of linear ACF	X	X
2	Heavy tails	X	X
3	Gain/Loss asymmetry		
4	Aggregational Gaussianity	X	X
5	Intermittency	X	X
6	Volatility Clustering	X	X
7	Conditional heavy tails	X	X
8	Slow decay of abs. ACF	∼	X
9	Leverage effect		
10	Volume/volatility corr.	X	
11	Asymmetry in timescales	∼	

Nuances to these results are unpacked in detail in the below subsections.

A. Linear Autocorrelation of Returns

Stylized Fact #1 expects linear autocorrelation in returns to be "insignificant, except for very small intraday timescales ($\approx$ 20 minutes) for which microstructure effects come into play." [15]. The linear autocorrelation function (ACF) is $C(\tau, \Delta t) = \text{corr}(r(t, \Delta t), r(t + \tau, \Delta t))$. Shown in Fig. 2, linear autocorrelation of the 1-minute returns is found to be rather weak and difficult to differentiate from white noise past lags of about eight minutes. The first-lag ACF values are negative and outside the range of the white noise returns. Past the first lag, the sign of the ACF varies by symbol. The magnitudes of the correlations go to zero, although not all are within the range of white noise until roughly the ninth lag.

At the trade-level, we see negative ACF values in the first lag, characteristic of the so-called 'bid-ask bounce' [3] [15]. Starting between -0.25 and -0.5 at the first lag, the ACF goes to zero within the next few lags. By lag $\tau = 4$, at least some of the symbols have positive ACF while others are negative. Due to the large number of observations in this timescale, the white noise levels are very small, and the observed values fall outside those thresholds. Given this, we rely on the fact that the sign of the ACF varies by the symbol and is relatively small (below 0.02 for $\tau > 10$) to argue that linear dependence is unpredictable past the first lag and weak past the first ten lags.

B. Heavy Tails and Aggregational Gaussianity

Cont's 2^{nd} fact expects return distributions to exhibit heavy tails. As done by Cont in [15] [16] [18], we examine the fourth central moment of returns, kurtosis. The kurtosis provides a measure of the tailedness of a distribution. We calculate kurtosis as defined below:

$$K(\Delta t) = \frac{\langle (r(t, \Delta t) - \langle r(t, \Delta t) \rangle)^4 \rangle}{\sigma(\Delta t)^4} - 3,$$

where $\sigma(\Delta t)^2$ is the variance of the returns. Note that this definition of kurtosis subtracts 3 in order for the normal distribution to have a kurtosis of zero. Positive kurtosis therefore means a distribution displays a sharper peak and heavier tails

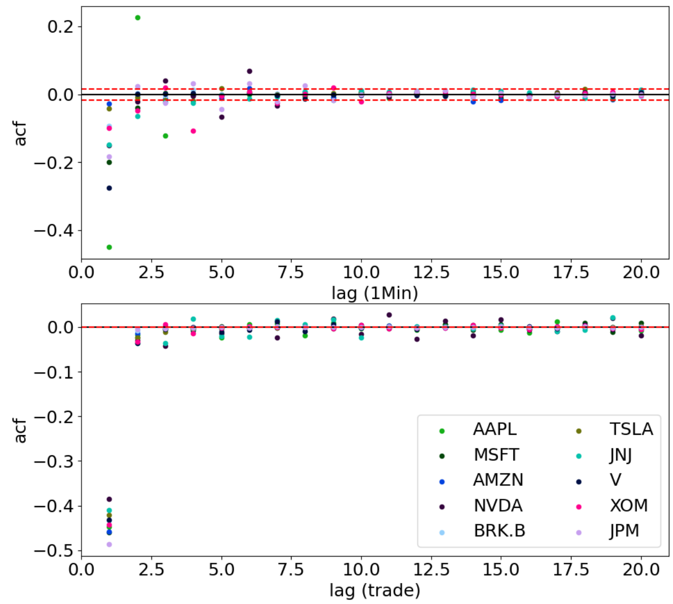


Fig. 2: Linear autocorrelation of returns.

than a normal distribution. Similar to Cont et al. [18] [42], we plot the kurtosis as a function of $j\Delta t$, with the expectation being for $K(j\Delta t)$ to be positive but decreasing as j increases.

We see the expected excess kurtosis in clock-time and event-time, as shown in Fig. 3. For $\Delta t = 1Min$, the kurtosis values range in magnitude from 10 to 10^3 depending on the symbol, with an overall negative trend as the timescale increases. The exception to this trend is the symbol JNJ, whose kurtosis stays higher than the other symbols and is actually higher for $\Delta t = 60Min$ than $\Delta t = 1Min$. All symbols stay above the range of the Gaussian white noise returns, as shown by the red line in the plots. Both these features are as expected by Fact #4, 'aggregational Gaussianity', with past results finding return kurtosis to decrease with timescale but stay positive for timescales of up to multiple days [39] [18] [42].

The event-time returns exhibit heavy tails and aggregational Gaussianity even more clearly than clock-time. The kurtosis of the trade-level returns ranges from around 10^2 to more than 10^6 depending on the symbol. Through aggregation in event-time, we see kurtosis decrease in a nearly monotonic fashion, going below $K(N) = 1$ and nearing the levels of Gaussian white noise for $N \geq 2500$. In clock-time, there are more than 2500 trades in fewer than 30-minutes for even our lowest-traded stock (BRK.B, Table II), and $\Delta t = 15Min$ returns in clock-time still exhibit heavy tails, as discussed above. Volume provides a proxy for volatility (as discussed later for Fact #10 in Section IV-G), and as such viewing returns in event-time is one method of correcting for volatility in the return series. Through this method of volatility correction, we see the return distributions converge more quickly to the normal distribution as a function of timescale.

This latter finding is in keeping with Cont's 7^{th} stylized fact, conditional heavy tails. We can further test this property by normalizing the returns by their mean and variance on a

TABLE II: Stats on the number of trades in a minute.

Symbol	Mean	Variance	Kurtosis
AAPL	648.06	366851.26	102.90
MSFT	603.67	272404.60	50.42
AMZN	391.88	127703.16	53.59
NVDA	369.12	121236.91	56.33
BRK.B	99.83	6606.73	57.60
TSLA	233.00	67692.21	56.37
JNJ	165.32	30041.24	79.54
JPM	281.42	45062.16	64.54
V	192.65	22253.93	53.18
XOM	210.46	29874.76	102.19

daily basis. More specifically, let T denote the trading day t is in. We then define the normalized returns as:

$$\hat{r}(t, \Delta t) = \frac{r(t, \Delta t) - \mu(T, \Delta t)}{\sigma(T, \Delta t)},$$

where $\mu(T, \Delta t) = \langle r(t \in T, \Delta t) \rangle$ and $\sigma(T, \Delta t) = \langle (r(t \in T, \Delta t) - \mu(T, \Delta t))^2 \rangle$. The kurtosis for the normalized returns as a function of timescale is shown in Fig. 4. We see that daily normalization does reduce the kurtosis from the unconditional returns while still leaving some excess kurtosis at small timescales. This is exactly as expected from Cont's description of conditional heavy tails. As we increase the timescales, both the calendar- and event-time kurtosis values go to zero and even slightly negative. The average kurtosis of the normalized returns is within the range of Gaussian white noise for $\Delta t \geq 15Min$ (for comparison, Andersen found similar at 5-minute timescale in the 1990's [2]).

C. Gain/Loss Asymmetry

Cont details his 3^{rd} stylized fact (gain/loss asymmetry) as prices experiencing larger drawdowns than upward movements. As mentioned in Section II, it is not exactly clear

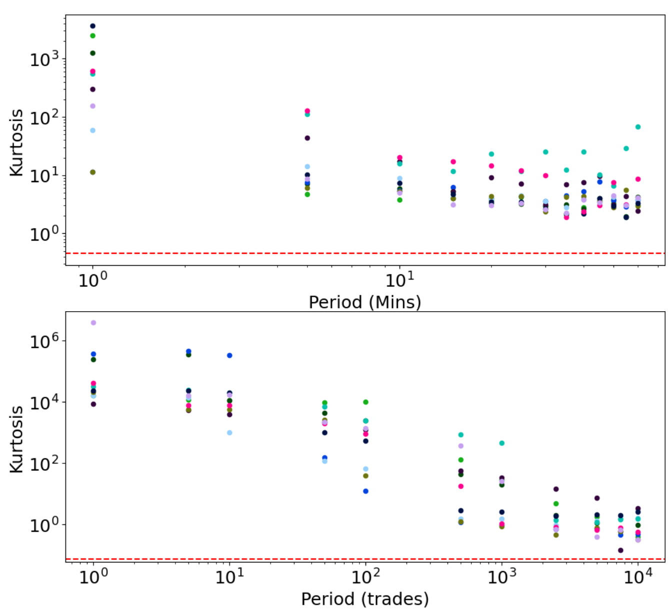


Fig. 3: Kurtosis as a function of timescale.

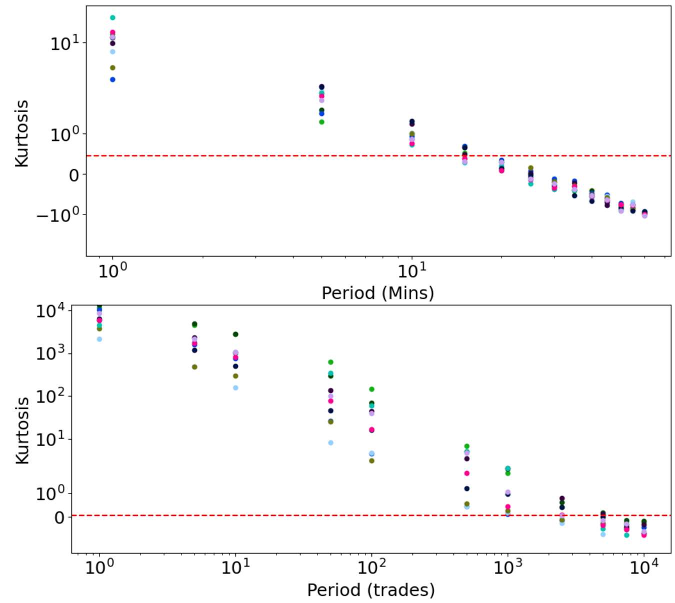


Fig. 4: Kurtosis of returns normalized by the daily variance.

what results were being referenced for this fact. Under some interpretations, returns should be expected to show negative skews, with skew measured as:

$$S(\Delta t) = \frac{\langle (r(t, \Delta t) - \langle r(t, \Delta t) \rangle)^3 \rangle}{\sigma(\Delta t)^3}.$$

We do not see this property consistently across symbols for 1-minute or trade-level returns, as shown in Table III.

We considered a more literal read of Cont's details for this fact as well, from which we would expect to see larger losses than we do gains. We measured the percentage of returns that are negative versus positive for different cutoffs. For a given timescale and quantile q, the cutoff is that quantile of a symbol's absolute returns in the timescale. The expectation would be for most of the extreme returns to be losses and for this to be more true as the quantile-cutoff gets closer to the 100^{th} percentile. We instead found more than half of the symbols having more extreme gains than losses for most quantiles[3]. We therefore do not find evidence of a gain/loss asymmetry effect in clock-time or event-time in our data.

[3] Shown in our supplementary material

TABLE III: Skew of 1-minute and trade-level returns.

Symbol	1-minute Skew	Trade-level Skew
AAPL	0.32	0.01
MSFT	0.32	0.00
AMZN	-0.21	0.63
NVDA	0.21	-0.01
BRK.B	-0.13	-0.17
TSLA	-0.24	-0.17
JNJ	1.32	0.10
V	0.37	0.12
XOM	0.35	0.18
JPM	-0.22	-0.00

D. Volatility Clustering

Cont's 6^{th} stylized fact expects volatility to cluster in time. We calculate volatility clustering by looking at the autocorrelation of absolute returns. Similar to linear ACF, let $C^0(\tau) = \text{corr}(|r(t, \Delta t)|, |r(t+\tau, \Delta t)|)$. The expectation is for $C^0(\tau, \Delta t) > |C(\tau)|$ and for $C^0(\tau)$ to asymptotically go to zero, with a decay that looks roughly linear on a log-log plot. This latter property, a power-law decay of autocorrelation, is claimed in the details of Fact #8. As shown in Fig. 5, absolute ACF of 1-minute returns starts above the values we saw for linear autocorrelation (Section IV-A) and remain consistently outside the range of white noise for all 300 lags tested. In log-log time, the shape of the decay appears to be sub-linear, although this path is noisier for some symbols than others. A couple symbols have absolute ACF near the levels of white noise by the last lags.

In trade-time, the absolute ACF is around the same level as the 1-minute returns on average but with a different shape in the subsequent decays of this effect. The rate of decay appears to slow after the first few lags and level out for more than 1000 lags before either continuing or hitting an exponential cutoff. After about 3000 lags, a couple stocks dip below the white-noise line while others stay above 0.01. An exponential cutoff occurs for a few stocks at differing places in the later lags. The overall path suggests a slower than exponential decay, or 'long-memory' of volatility in trade-time. Note, however, that even though the clock-time results show a sub-linear trend in log-log scale they stay higher on average than the event-time results for the lags tested[4].

[4]There are also more than 10,000 trades in 200 minutes on average (Table II)

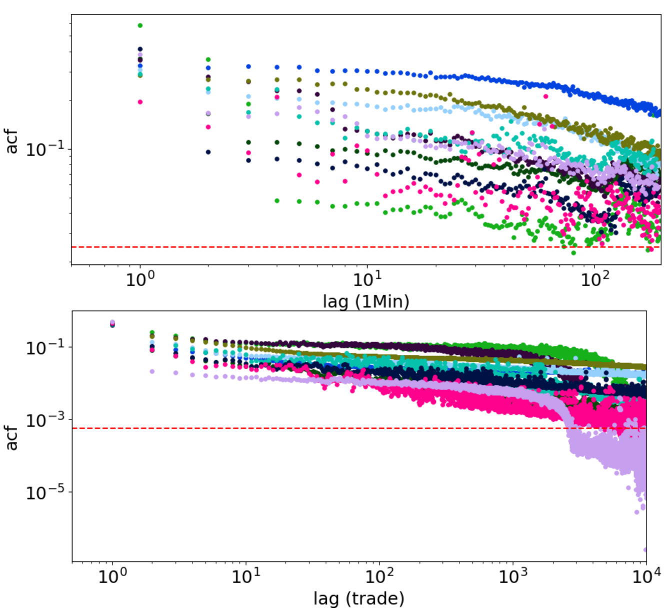


Fig. 5: Autocorrelation of absolute returns.

E. Intermittency

As detailed in Section II, the property of intermittency follows from and is intimately tied to volatility clustering and heavy tails in the literature. We attempt one additional measure of intermittency, however, through examining the distribution of interarrival times of extreme price moves. Specifically, consider the 99^{th}-percentile biggest absolute returns for a given symbol, which we will denote $N_{0.99,\Delta t}$. We can then count the number of these returns we see in a given period of time, with greater variability providing a measure of intermittency. We show these extreme returns occur more variably than they would if arising from a Poisson distribution by measuring their Fano factor. The Fano factor is defined as

$$F(\Delta t) = \frac{\sigma^2_{N_{0.99,\Delta t}}}{\langle N_{0.99,\Delta t} \rangle},$$

the ratio of the variance to the mean for the number of extreme returns in a period. This ratio would be 1 for a Poisson distribution, but we see this is not the case in Table IV. The Fano factor is greater than one for extreme trade-level returns in 1-minute periods as well as the extreme 1-minute returns in 30-minute periods. We furthermore found the distribution of interarrival times between intraday extreme returns to show excess kurtosis[5]. From these findings, along with the heavy-tails and volatility clustering already discussed, we see evidence of intermittency in clock-time and event-time.

F. Leverage Effect

Cont's 9^{th} stylized fact asserts that volatility is negatively correlated with the returns for an asset. We measure the leverage effect (Fact #9) as Cont laid out in [15], drawing from the results of Bouchaud et al. [12] and Pagan [40]. This approach looks simply at the correlation between returns and lagged volatility, with volatility measured as the squared returns: $L(\tau, \Delta t) = \text{corr}\left(|r(t+\tau, \Delta t)|^2, r(t, \Delta(t))\right)$. The expectation is for $L(\tau)$ to be negative for $\tau = 1$ and to be larger with positive τ than for the corresponding $-\tau$.

We find no clear trend to the correlation values across symbols. There are varying strengths and signs to the correlations at each lag, suggesting they might arise from specific

[5]Shown in our supplementary material

TABLE IV: Fano factor of extreme returns. For returns with magnitudes in the 99^{th}-quantile, we measured the Fano factor of the number of extreme returns in a coarse period of time. For 1-minute returns, the coarse period was 30-minutes, for trade-level returns, the coarse period was 1-minute.

Symbol	1-min Returns	Trade Returns
AAPL	2.11	311.38
MSFT	2.46	126.51
AMZN	2.99	108.11
NVDA	3.50	102.21
BRK.B	1.98	15.79
TSLA	3.55	67.16
JNJ	3.76	53.90
JPM	2.56	51.75
V	2.55	62.21
XOM	2.12	26.78

variation in the path of a given symbol's price over our observation period. For some symbols, there is an interesting symmetry where $L(-1) \approx -L(1)$. This also goes against the descriptions from Cont [15] and Bouchaud et al. [12], who described the relationship between returns and negatively lagged volatility as being largely insignificant. Overall, we do not see the expected direction of the leverage effect relationship, nor do we see any clear trend in this relationship across the symbols.

G. Volume/Volatility Correlation

The relationship between volume and volatility is Cont's 10^{th} stylized fact and asserts a positive correlation between volume and volatility. We examine this by taking the correlations of the volume of shares[6] in a period with the lagged absolute returns. Indeed, in clock-time we do see a strong, persistent correlation between shares traded and volatility for each of the symbols. We see the correlation values stay above the correlation seen for white noise returns for most positive and negative lags tested. In other words, volume correlates strongly with lagged volatility and vice versa. In event-time, most symbols show a weak relationship between share volume and volatility, synchronously and at a lag of $\tau = 1$. At all other lags (including negative lags) the relationship is roughly zero for all symbols. JPM has a much stronger relationship at both of these lags, but it also has virtually no correlation at any lag besides $\tau = 0$ or 1. These results give some explanation towards the differences in how trade-level and clock-time returns behave.

[6]In clock-time, correlation between the volume of trades and volatility was found to be very similar to that between shares and volatility.

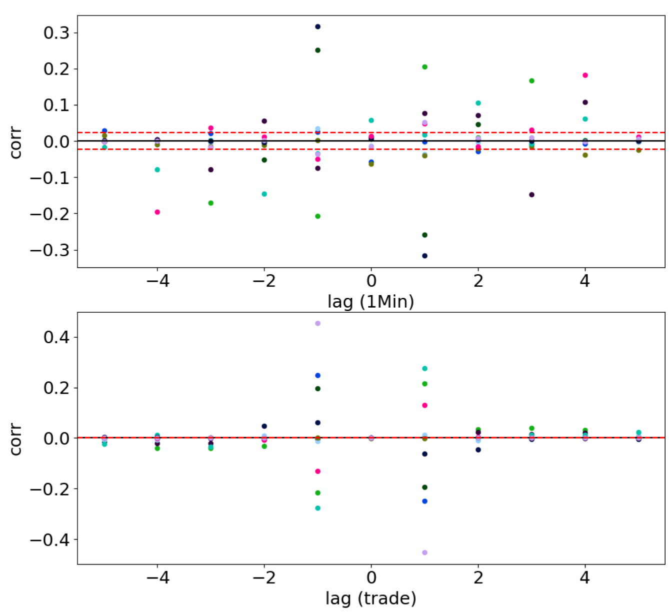


Fig. 6: Leverage effect, measured as the correlation between $r(t, \Delta t)$ and $r(t+\tau, \Delta t)^2$.

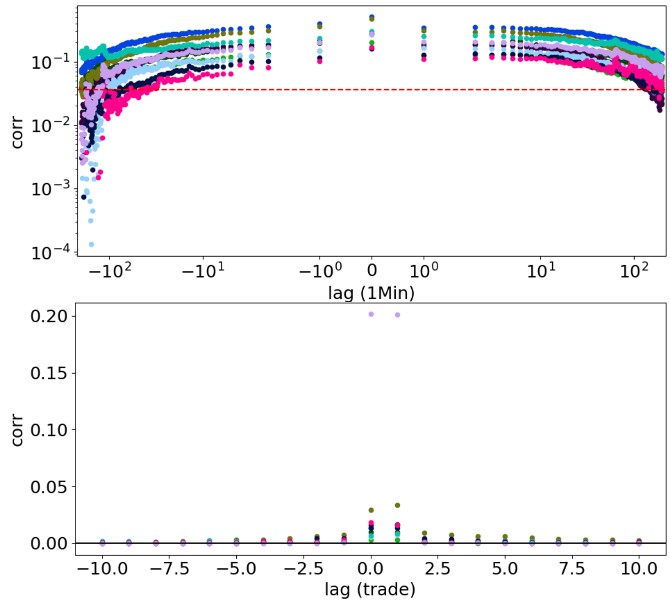


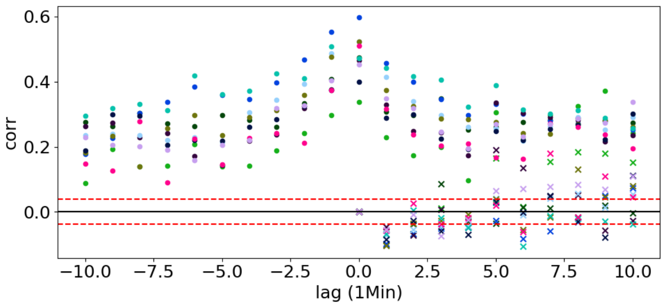
Fig. 7: Share volume versus lagged volatility.

H. Asymmetry in Timescales

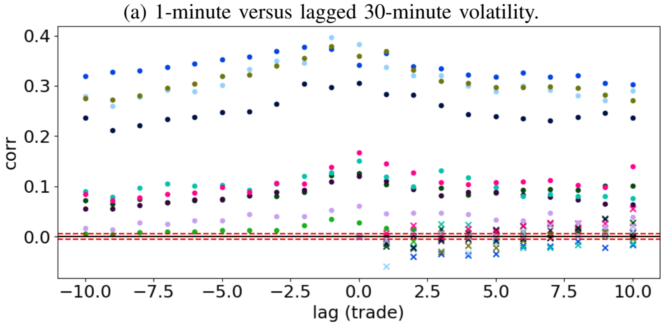
The final stylized fact examines the asymmetry of the flow of information across timescales. In clock-time, we consider $\Delta t = 1Min$ and $\Delta T = 30Min$ as our fine and coarse timescales, respectively. In event-time, we use trade-time as Δt and $N = 1000$ as our coarse timescale ΔT. We calculate $A(\tau) = \text{corr}\left(\overline{|r(t \in T, \Delta T)|}, |r(T+\tau, \Delta T)|\right)$, as done by Müller et al. [38]. We measure the asymmetry by differencing the correlation at the corresponding positive and negative lags τ. The expectation is for correlation at the negative lags to be larger than at the corresponding positive lags for at least a few steps.

We first of all see a strong relationship between coarse-grained volatility and fine-grained volatility, synchronously and at a lag, for each timescale and perspective (Fig. 8). As in Müller et al. [38], the relationship is strongest at lag $\tau = 0$. Comparing positive to negative lags gives an indication of a possible causal relationship, and we do see a slight edge given to the negative lags over the positive lags in clock-time. The smallest values of $A(1, \Delta t) - A(-1, \Delta t)$ are around the range seen from white noise, however, which must be kept in mind despite the consistent signal seen across symbols. After the first lag, the effect is approximately zero, with varying asymmetries depending on symbol, lag, and perspective. The asymmetry effect is not found in event-time, with the difference $A(1, N) - A(-1, N)$ being positive or negative depending on the symbol. Furthermore, when we tested clock-time results with 10-minutes as the coarse timescale, we found no consistent asymmetry across symbols[7]. We therefore only have a consistent trend for one lag in one timescale, calling into question the strength of this effect.

[7]Given in our supplementary materials



(a) 1-minute versus lagged 30-minute volatility.



(b) Trade versus lagged 1000-trade volatility.

Fig. 8: Average fine-grained volatility versus lagged coarse-grained volatility.

I. Discussion

As summarized in Table I, we have shown evidence for eight of Cont's 11 stylized facts holding true in general for each of the 10 stocks tested, with weak evidence in support of one other fact. Some of these facts are better expressed in clock-time than event-time, however, and vice versa. We show correlation between volume and volatility (Fact #10) in clock-time, with limited to no support found in the trade-level returns. Only weak results are found for asymmetry in timescales (Fact #11), with slight asymmetry only found for one timescale. Trade-level returns show a slower decay of absolute autocorrelation than 1-minute returns, more in keeping with the power-law decay claimed by Fact #8. Aggregational Gaussianity (Fact #4) is also more clearly expressed in event-time than clock-time, although the property is overall expressed in both clocks.

The measures examined for Facts #3 and #9 do not provide evidence of gain/loss asymmetry or the leverage effect being present in general for individual stocks. In contrast to the claims of these facts, skew and correlation of returns with volatility were both found to vary in sign and strength depending on the symbol and timescale. The signs of the largest price moves were also not found to tend negative in our sample. Past research using measures that look at the motion of multiple stocks together (such as an index or the average over multiple stocks) have found support for these facts, however [12] [27] [28] [46]. Future research could look at whether these facts might hold for groups of stocks (e.g. indices) collectively in modern markets. Nevertheless, our results indicate that an individual stocks should not be assumed to express these properties in general.

Finally, the analysis here focuses on the SIP feeds inclusive of trades that occurred on stock exchanges and in other venues. Off-exchange trades were found to lead to some of the noisy signals in our results, raising the question of how differently they behave from the trades occurring on exchanges. Future work could examine the off-exchange trades to test whether they exhibit the same set of stylized facts on their own

V. Conclusion

Cont's original set of stylized facts [15] emerged from a synthesis of empirical studies, each study focused on a market which existed prior to 2001. As demonstrated elsewhere in previous studies, the technological arms race and resulting market fragmentation in the intervening decades since Cont's study fundamentally changed the dynamics of the U.S. stock market [29] [47]. Motivated by these market changes to revisit Cont's original study, we find strong evidence for eight of Cont's original set of 11 stylized facts. A robust set of stylized facts serves at least two distinct communities. For the community of financial regulators, the set of stylized facts provides guideposts against which to assess the impacts of regulatory reform, both the intended and unintended impacts. For the scientific community, the set of stylized facts provides the guideposts for the design, development, test and calibration for the next generation of market models.

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